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On connected line sets of antiflag class $[0, \alpha, q]$ in $\mathrm{AG}(n, q)$

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Abstract

We investigate the partial linear spaces, fully embedded in an affine space $\mathrm{AG}(n, q)$ with the property that for every antiflag $\{p, L\}$, the number of lines through p intersecting L is either 0, α, or q. Besides some general results we prove a complete classification of those geometries fully embedded in an affine plane of order q and of the connected geometries with $1 < \alpha < q$, fully embedded in $\mathrm{AG}(3, q)$.

1. Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *partial linear space* if every two points are incident with at most one line. It is said to be of order (s, t), if every line is incident with $s + 1$ points, while every point is on $t + 1$ lines. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be of *antiflag class* $[\alpha_1, \ldots, \alpha_m]$ if for every antiflag $\{p, L\}$, i.e. for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, not through p, the so-called *incidence number* $\alpha(p, L)$ of lines of $\mathcal{L}$ through p which intersect L is one of $\alpha_1, \ldots, \alpha_m$. We do not require that every incidence number actually occurs. If they do all occur then we say that the partial linear space is of *antiflag type* $(\alpha_1, \ldots, \alpha_m)$. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *fully embedded* in an affine space $\mathrm{AG}(n, q)$, also called *an affine partial linear space*, if $\mathcal{L}$ is a set of lines of $\mathrm{AG}(n, q)$, $\mathcal{P}$ is the set of all affine points on the lines of $\mathcal{S}$ and I is the incidence of $\mathrm{AG}(n, q)$. We also require that $\mathcal{P}$ spans $\mathrm{AG}(n, q)$.

Our goal is to investigate the connected affine partial linear spaces of antiflag class $[0, \alpha, q]$. Actually, as every affine partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is defined by its line set $\mathcal{L}$, the study of affine partial linear spaces is equivalent to the study of subsets of the line set of $\mathrm{AG}(n, q)$. We

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say that $\mathcal{L}$ is *a connected line set* if $\mathcal{S}(\mathcal{L})$ is connected, and we will say that $\mathcal{L}$ is of antiflag class $[\alpha_1, \ldots, \alpha_m]$ (or antiflag type $(\alpha_1, \ldots, \alpha_m)$) if $\mathcal{S}(\mathcal{L})$ has this property.

We have the following interesting example in mind. Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in a finite projective space $\mathrm{PG}(n+1, q)$, $n \geq 3$. Consider a point $r \notin \mathcal{Q}_{n+1}$, distinct from its nucleus if $n+1$ and q are even, and a hyperplane $\mathrm{PG}(n, q)$ not through r. Let $\mathcal{R}_n$ be the projection of the quadric $\mathcal{Q}_{n+1}$ from the point r on the hyperplane $\mathrm{PG}(n, q)$; let $\mathcal{T}_n \subseteq \mathcal{R}_n$ be the set of points p of $\mathrm{PG}(n, q)$ such that the line $\langle p, r\rangle$ is a tangent to $\mathcal{Q}_{n+1}$ and let $\mathcal{P}_n = \mathcal{R}_n \setminus \mathcal{T}_n$. We denote by HT_n the partial linear space whose points are the elements of $\mathcal{P}_n$ and whose lines are the lines of $\mathrm{PG}(n, q)$ which contain q points of $\mathcal{P}_n$. It is easy to check that this geometry HT_n is of antiflag class $[0, 2, q]$. Moreover, if q is even, $\mathcal{T}_n$ is the set of points of a hyperplane Π_∞ of $\mathrm{PG}(n, q)$, hence the geometry HT_n is an affine partial linear space. If n is even, we write HT_n^+ if $\mathcal{Q}_{n+1}$ is a nonsingular hyperbolic quadric, and HT_n^- if $\mathcal{Q}_{n+1}$ is a nonsingular elliptic quadric.

2. Line sets of antiflag type $[0, \alpha, q]$ in an affine plane of order q

Clearly, a line set $\mathcal{L}$ of $\mathrm{AG}(n, q)$ is of antiflag class $[0, \alpha, q]$ if and only if for every affine plane π, $\mathcal{L}_\pi$ is of antiflag class $[0, \alpha, q]$. Hence it is useful to investigate first the partial linear spaces of antiflag class $[0, \alpha, q]$ embedded in an affine plane $\mathrm{AG}(2, q)$. In the next theorem we give a complete classification for the planar case.

Theorem 2.1. *If $\mathcal{L}$ is a nonempty line set of antiflag class $[0, \alpha, q]$ in an affine plane $\mathrm{AG}(2, q)$, $0 < \alpha < q$, then one of the following cases occurs.*

1. *$\mathcal{L}$ consists of a number of parallel lines, hence $\mathcal{L}$ is of antiflag type (0).*
2. *$\mathcal{L}$ consists of $\alpha + 1$ parallel classes of lines, hence $\mathcal{L}$ is of antiflag type (α). We say that $\mathcal{L}$ is a planar net of order q and degree $\alpha + 1$.*
3. *$\alpha = 2$, $q = 2^h$, and $\mathcal{L}$ is an oval in the dual plane of $\mathrm{AG}(2, q)$, the nucleus being the line at infinity, hence $\mathcal{L}$ is of antiflag type $(0, 2)$. We say that $\mathcal{L}$ is a dual oval.*
4. *$\mathcal{L}$ consists of all lines of $\mathrm{AG}(2, q)$, hence $\mathcal{L}$ is of antiflag type (q).*
5. *$\alpha = 1$ and $\mathcal{L}$ only contains lines of two parallel classes of lines, hence $\mathcal{L}$ is of antiflag type $(0, 1)$.*
6. *$\alpha = 1$ and $\mathcal{L}$ only contains a number of lines through a given affine point, hence $\mathcal{L}$ is of antiflag type (1).*
7. *$\alpha = q - 1$ and $\mathcal{L}$ contains all lines of $\mathrm{AG}(2, q)$ except a number of parallel lines, hence $\mathcal{L}$ is of antiflag type $(q - 1, q)$.*
8. *$\alpha = q - 1$ and $\mathcal{L}$ contains all lines of $\mathrm{AG}(2, q)$ except a number of lines through a given affine point, hence $\mathcal{L}$ is of antiflag type $(q - 1, q)$.*

Proof. If $\mathcal{L}$ is of antiflag class $[0, \alpha]$ with $\alpha > 1$, then it is shown in [4] that we are in case 1, 2 or 3. It is straightforward to show that line sets of antiflag class $[0, 1, q]$ in an affine plane $\mathrm{AG}(2, q)$ are necessarily the ones described in 1, 2, 4, 5 and 6.

So we may assume that $q > 2$ and that $\mathcal{L}$ is a line set in $\mathrm{AG}(2, q)$ of antiflag class $[0, \alpha, q]$ with $1 < \alpha < q$ and such that there is a point p and a line $L \in \mathcal{L}$ not through p such that $\alpha(p, L) = q$. Then every parallel class of lines contains at least one line of $\mathcal{L}$.

For every affine point r, let $t_r + 1$ be the number of lines of $\mathcal{L}$ through r. Let L' be the line through p parallel to L, and let $r \neq p$ be an affine point of L'. Suppose that $0 < t_r + 1 < q + 1$.

As $0 < t_r + 1$, there is a line $M \in \mathcal{L}$ through r. Let $M' \neq M$ be a line of $\mathcal{L}$ parallel to M. Then $\alpha(r, M') = t_r$. As $t_r + 1 < q + 1$, $t_r + 1 \in \{1, \alpha + 1\}$.

Since $t_r + 1 < q + 1$, there is a line $N \notin \mathcal{L}$ through r. Let $N' \neq N$ be a line of $\mathcal{L}$ parallel to N. Then $\alpha(r, N') = t_r + 1$. As $0 < t_r + 1$, $t_r + 1 \in \{\alpha, q\}$. Since $1 < \alpha < q$, we may conclude that $t_r + 1 = q = \alpha + 1$. So $\alpha = q - 1$.

Suppose that $1 < \alpha < q - 1$. Then for every affine point $r \neq p$ of L', $t_r + 1$ is either 0 or $q + 1$. Suppose that $t_r + 1 = q + 1$ for some point $r \neq p$ of L'. Then $L' \in \mathcal{L}$ and hence every affine point $r' \in L$ has $t_{r'} + 1 = q + 1$, so we are in case 4. Suppose that $t_r + 1 = 0$ for every point $r \neq p$ of L'. Then the only lines of $\mathcal{L}$ which are not parallel to L are the lines through p intersecting L in an affine point. Let r be an affine point not on L or L'. Then the line $\langle p, r\rangle$ is the only line of $\mathcal{L}$ through r intersecting L. So $\alpha(r, L) = 1$, a contradiction.

Suppose that $\alpha = q - 1$. If $q = 3$, one can easily show that only cases 7 and 8 occur. Suppose that $q > 3$ and that r is an affine point on a line $M \in \mathcal{L}$. Clearly there is a line $M' \in \mathcal{L}$ not containing r which intersects M in an affine point. Hence $\alpha(r, M') \geq q - 1$, so $t_r + 1 \geq q - 1$. Suppose there is an affine point r such that $t_r + 1 = q - 1$ and let L_1, L_2 be the lines through r not in $\mathcal{L}$. It is easy to show (note that $q > 3$) that there is a line $L_3 \in \mathcal{L}$ not through r which intersects both L_1 and L_2 in an affine point. We conclude that if $q > 3$, then every affine point r has $t_r + 1 \in \{0, q, q + 1\}$. It follows that if $q > 3$, through every affine point there are 0, 1 or $q + 1$ lines which are not in $\mathcal{L}$. And so we are in case 7 or 8. □

3. Some general results on affine line sets of antiflag class $[0, \alpha, q]$

If $\mathcal{S}(\mathcal{L})$ is an affine partial linear space, then for every affine subspace U, let $\mathcal{L}_U$ be the set of lines of $\mathcal{L}$ in U, and let $\mathcal{S}(\mathcal{L})_U$ be the partial linear space $(\mathcal{P}_U, \mathcal{L}_U, \mathrm{I}_U)$, where $\mathcal{P}_U = \mathcal{P} \cap U$ and I_U is the incidence I restricted to $\mathcal{P}_U$ and $\mathcal{L}_U$. Note that $\mathcal{S}(\mathcal{L})_U$ is not the same as $\mathcal{S}(\mathcal{L}_U)$, as $\mathcal{S}(\mathcal{L})_U$ may contain isolated points, that is, points that are on no line, which is not the case for $\mathcal{S}(\mathcal{L}_U)$.

Highly irregular examples of connected line sets of antiflag class $[0, 1, q]$ or $[0, q - 1, q]$ in $\mathrm{AG}(n, q)$ exist and are easy to construct. Therefore we restrict our attention from now on to connected line sets of antiflag class $[0, \alpha, q]$ in $\mathrm{AG}(n, q)$, with $1 < \alpha < q - 1$.

Lemma 3.1. *Let $\mathcal{L}$ be a line set of $\mathrm{AG}(n, q)$, $n \geq 3$, of antiflag class $[0, \alpha, q]$, with $1 < \alpha < q - 1$. Let π be an affine plane. Then one of the following cases occurs.*

Type I. *π does not contain any line of $\mathcal{L}$.*
Type II. *π only contains a number of parallel lines of $\mathcal{L}$.*
Type III. *$\mathcal{L}_\pi$ is a planar net of order q and degree $\alpha + 1$.*
Type IV. *$\alpha = 2$, $q = 2^h$ and $\mathcal{L}_\pi$ is a dual oval.*
Type V. *$\mathcal{L}_\pi$ consists of all lines of π.*

Proof. This follows immediately from Theorem 2.1. □

Let Π_∞ denote the space at infinity of $\mathrm{AG}(n, q)$. For every point p on an element of a line set $\mathcal{L}$ of $\mathrm{AG}(n, q)$, let θ_p denote the set of points at infinity of the lines of $\mathcal{L}$ through p.

Corollary 3.2. *Let $\mathcal{L}$ be a connected line set of $\mathrm{AG}(n, q)$, $n \geq 3$, of antiflag class $[0, \alpha, q]$, with $1 < \alpha < q - 1$. Then there exists a positive integer t such that $\mathcal{S}(\mathcal{L})$ is of order $(q - 1, t)$. If U is an affine subspace of dimension $m \geq 2$, then every connected component $\mathcal{L}'$ of $\mathcal{L}_U$ is of antiflag class $[0, \alpha, q]$, whence $\mathcal{S}(\mathcal{L}')$ has an order. If the lines of $\mathcal{L}$ span $\mathrm{AG}(n, q)$, then for every point p of $\mathcal{S}(\mathcal{L})$, the set θ_p spans Π_∞.*

Proof. Let p and r be distinct affine points on a line $L \in \mathcal{L}$. It follows from Lemma 3.1 that in every plane π through L, the numbers of lines of $\mathcal{L}$ through p and r are equal. Hence p and r are on equally many lines of $\mathcal{L}$. By connectedness, it follows that $\mathcal{S}(\mathcal{L})$ has an order.

The second statement is straightforward.

Suppose that $\mathcal{L}$ spans $\mathrm{AG}(n, q)$. Let p be a point of $\mathcal{S}(\mathcal{L})$, and suppose that θ_p is contained in a proper subspace U_∞ of Π_∞. Let $U = \langle p, U_\infty \rangle$, and let $\mathcal{L}'$ be the connected component of $\mathcal{L}_U$ which contains the lines of $\mathcal{L}_U$ through p. Since $\theta_p \subseteq U_\infty$, every line of $\mathcal{L}$ through p is contained in U, and hence in $\mathcal{L}'$. It follows that the order of $\mathcal{S}(\mathcal{L}')$ is the same as the order of $\mathcal{S}(\mathcal{L})$. So for every affine point p' on a line of $\mathcal{L}'$, every line of $\mathcal{L}$ through p' is in $\mathcal{L}'$. Since $\mathcal{L}$ is connected, this yields $\mathcal{L}' = \mathcal{L}$. So $\mathcal{L}$ is contained in U, a proper subspace of $\mathrm{AG}(n, q)$, a contradiction. □

4. Singular line sets of antiflag class $[0, \alpha, q]$

A line set $\mathcal{L}$ of $\mathrm{AG}(n, q)$, or the affine partial linear space $\mathcal{S}(\mathcal{L})$, of antiflag class $[0, \alpha, q]$ is said to be *singular* if there exists a point p_∞ in the hyperplane at infinity Π_∞ such that the following conditions hold.

1. If p is a point of $\mathcal{S}(\mathcal{L})$, then the line $\langle p, p_\infty \rangle$ is in $\mathcal{L}$.
2. If $L \in \mathcal{L}$ and L does not intersect Π_∞ in the point p_∞, then the plane $\langle L, p_\infty \rangle$ is of type V.

We will call every such point p_∞, a *singular point*. Notice that the first condition is superfluous if $\mathcal{L}$ does not contain any isolated lines (that is, connected components consisting of a single line).

An example of a singular line set of antiflag class $[0, \alpha, q]$ is the following. Let $\mathcal{L}'$ be a nonsingular line set of antiflag class $[0, \alpha, q]$ in a subspace U of $\mathrm{AG}(n, q)$ of dimension $2 \le m \le n-1$, and let V_∞ be a subspace of Π_∞ of dimension $n-m-1$, skew to $U \cap \Pi_\infty$. Then the line set $\mathcal{L}$ consisting of all the affine lines in the $(n-m+1)$-spaces $\langle L, V_\infty \rangle$, with $L \in \mathcal{L}'$, is a singular line set of $\mathrm{AG}(n, q)$ of antiflag class $[0, \alpha, q]$. The singular points are precisely the points of V_∞. The set $\mathcal{L}$ is connected if and only if $\mathcal{L}'$ is connected; the lines of $\mathcal{L}$ span $\mathrm{AG}(n, q)$ if and only if the lines of $\mathcal{L}'$ span U. We say that $\mathcal{L}$ is *the singular line set with vertex* V_∞ *and base* $\mathcal{L}'$.

Theorem 4.1. *Let $\mathcal{L}$ be a singular connected line set of* $\mathrm{AG}(n, q)$ *of antiflag class* $[0, \alpha, q]$, $1 < \alpha < q-1$, *such that the lines of $\mathcal{L}$ span* $\mathrm{AG}(n, q)$. *Then either $\mathcal{L}$ is the set of all affine lines, or it is the singular line set with vertex an $(n-m-1)$-space $V_\infty \subseteq \Pi_\infty$ and base a nonsingular connected line set $\mathcal{L}'$ of antiflag class* $[0, \alpha, q]$ *in an m-space U of* $\mathrm{AG}(n, q)$, *such that the lines of $\mathcal{L}'$ span U, U and V_∞ are disjoint, and* $2 \le m \le n-1$.

Proof. Let S_∞ denote the set of singular points of $\mathcal{L}$. We prove that S_∞ is the point set of a subspace of Π_∞. Suppose that p_∞ and p'_∞ are distinct points of S_∞, let $L_\infty = \langle p_\infty, p'_\infty \rangle$ and let $L \in \mathcal{L}$. First suppose that $L \cap \Pi_\infty \notin L_\infty$. Let W be the 3-space $\langle L, L_\infty \rangle$. As p_∞ and p'_∞ are singular points of $\mathcal{L}$, they are also singular points of $\mathcal{L}_W$. So every affine line of the planes $\pi = \langle L, p_\infty \rangle$ and $\pi' = \langle L, p'_\infty \rangle$ is a line of $\mathcal{L}_W$. Consequently, for all affine lines M and M' of the planes π and π' respectively, every affine line of the planes $\langle M, p'_\infty \rangle$ and $\langle M', p_\infty \rangle$ respectively, is a line of $\mathcal{L}_W$. Hence $\mathcal{L}_W$ is the set of all affine lines of W. In particular, for every point $p''_\infty \in L_\infty$, the plane $\pi'' = \langle L, p''_\infty \rangle$ is a plane of type V.

Now suppose that L intersects Π_∞ in a point of the line L_∞. Since p_∞ and p'_∞ are singular points, $\langle L, L_\infty \rangle$ is a plane of type V. It follows that every point p''_∞ on the line $L_\infty = \langle p_\infty, p'_\infty \rangle$ is a singular point of $\mathcal{L}$. Hence S_∞ is the point set of an $(n-m-1)$-space $V_\infty \subseteq \Pi_\infty$, $0 \le m \le n-1$.

If $m = 0$, then $\mathcal{L}$ is the set of all affine lines of $\mathrm{AG}(n, q)$. Suppose that $m = 1$. Then V_∞ is an $(n-2)$-space of Π_∞. As $\mathcal{L}$ is connected and not contained in a hyperplane, there is a line $L \in \mathcal{L}$ which intersects Π_∞ in a point $p_\infty \notin V_\infty$. As we have shown above, for every 3-space $W = \langle L, L_\infty \rangle$ with $L_\infty \subseteq V_\infty$, $\mathcal{L}_W$ is the set of all affine lines of W. It follows that $\mathcal{L}$ is the set of all affine lines of $\mathrm{AG}(n, q)$. So every point of Π_∞ is singular, a contradiction. Hence $m \neq 1$.

Suppose that $m \geq 2$. Let U be any affine m-space skew to V_∞. If $\mathcal{L}_U$ has a singular point $p_\infty \in U \cap \Pi_\infty$, then one easily proves that p_∞ is a singular point of $\mathcal{L}$. But $p_\infty \notin V_\infty$, a contradiction. So $\mathcal{L}_U$ is nonsingular. An affine line L which intersects Π_∞ in a point $p_\infty \notin V_\infty$ is in $\mathcal{L}$ if and only if the projection of L from V_∞ onto U is a line of $\mathcal{L}_U$. An affine line which intersects Π_∞ is a point $p_\infty \in V_\infty$ is in $\mathcal{L}$ if and only if the point $p = \langle L, V_\infty \rangle \cap U$ is a point of $\mathcal{S}(\mathcal{L}_U)$. It follows that $\mathcal{L}$ is the singular line set with vertex V_∞ and base $\mathcal{L}_U$. As $\mathcal{L}$ is connected, $\mathcal{L}_U$ is connected. As the lines of $\mathcal{L}$ span $\mathrm{AG}(n, q)$, the lines of $\mathcal{L}_U$ span U. As $\mathcal{L}$ is of antiflag class $[0, \alpha, q]$, $\mathcal{L}_U$ is of antiflag class $[0, \alpha, q]$. □

5. Linear representations

Let Π_∞ be a hyperplane of the projective space $\mathrm{PG}(n, q)$. The *linear representation* $T^*_{n-1}(\mathcal{K}_\infty)$ of a set $\mathcal{K}_\infty \subseteq \Pi_\infty$ is the affine partial linear space which has as line set the set of all affine lines of $\mathrm{AG}(n, q) = \mathrm{PG}(n, q) \setminus \Pi_\infty$ intersecting Π_∞ in a point of $\mathcal{K}_\infty$. It is easy to prove that $T^*_{n-1}(\mathcal{K}_\infty)$ is connected if and only if the set $\mathcal{K}_\infty$ spans Π_∞. The line set of $T^*_{n-1}(\mathcal{K}_\infty)$ is of antiflag class $[0, \alpha, q]$ if and only if $\mathcal{K}_\infty$ is a point set of class $[0, 1, \alpha + 1, q + 1]$, i.e., every line of Π_∞ intersects $\mathcal{K}_\infty$ in either $0, 1, \alpha + 1$, or $q + 1$ points. It is easy to see that $T^*_{n-1}(\mathcal{K}_\infty)$ has no planes of type IV. The converse also holds.

Theorem 5.1. *Let $\mathcal{L}$ be a connected line set of* $\mathrm{AG}(n, q)$ *of antiflag class* $[0, \alpha, q]$, *with* $1 < \alpha < q - 1$, *such that there are no planes of type* IV, *and such that the lines of $\mathcal{L}$ span* $\mathrm{AG}(n, q)$. *Then $\mathcal{L}$ is the line set of a linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ of a point set $\mathcal{K}_\infty$ of class* $[0, 1, \alpha + 1, q + 1]$.

Proof. If $n = 2$, the theorem holds by Lemma 3.1. Suppose that $n > 2$, and that the theorem holds for all $2 \leq m < n$.

Let $L_0 \in \mathcal{L}$. We prove that every affine line parallel to L_0 is in $\mathcal{L}$. Let p be an affine point of L_0 and let $p_\infty = L_0 \cap \Pi_\infty$. By Corollary 3.2, θ_p spans Π_∞. Hence there is an $(n-2)$-space $U_\infty \subseteq \Pi_\infty$ such that $p_\infty \in U_\infty$ and $\theta_p \cap U_\infty$ spans U_∞. Let $U = \langle L, U_\infty \rangle$ and let $\mathcal{L}'$ be the connected component of $\mathcal{L}_U$ which contains the line L. Then $\mathcal{L}'$ is a connected line set of U of antiflag class $[0, \alpha, q]$ such that there are no planes of type IV, and such that the lines of $\mathcal{L}'$ span U. By the induction hypothesis, $\mathcal{L}'$ is the line set of the linear representation of the set $\theta_p \cap U_\infty$. So every affine line of U, parallel to L_0, is a line of $\mathcal{L}$.

Since θ_p spans Π_∞, there is a line $M \in \mathcal{L}$ through p which intersects U in the point p. Let U' be a hyperplane parallel to but distinct from U, and let $p' = M \cap U'$. Let N be a line of $\mathcal{L}'$ through p, and let N' be the line through p' parallel to N. The plane $\pi = \langle M, N \rangle$ contains two intersecting lines of $\mathcal{L}$ and hence is of type III, IV or V. By assumption, π is not of type IV. Hence π is of type III or V, and so N' is a line of $\mathcal{L}$. Hence for every point $p'_\infty \in \theta_p \cap U_\infty$, the line $N' = \langle p', p'_\infty \rangle$ is in $\mathcal{L}$. Hence $\theta_p \cap U_\infty \subseteq \theta_{p'}$. Let $\mathcal{L}''$ be the connected component of $\mathcal{L}_{U'}$ containing the lines of $\mathcal{L}_{U'}$ through p'. Since $\theta_p \cap U_\infty$ spans U_∞, $\theta_{p'} \cap U_\infty$ spans U_∞, and so $\mathcal{L}''$ is a connected line set of U' of antiflag class $[0, \alpha, q]$ such that there are no planes of type IV, and such that the lines of $\mathcal{L}''$ span U'. By the induction hypothesis, $\mathcal{L}''$ is the line set of the linear representation of the set $\theta_{p'} \cap U_\infty$. As $p_\infty \in \theta_p \cap U_\infty \subseteq \theta_{p'} \cap U_\infty$, every line in U' which is parallel to L_0, is a line of $\mathcal{L}''$ and so of $\mathcal{L}$. Since this holds for every hyperplane U' parallel to U, every affine line parallel to L_0 is a line of $\mathcal{L}$. Hence $\mathcal{L}$ is a union of parallel classes of lines. Equivalently, $\mathcal{L}$ is the line set of a linear representation $T^*_{n-1}(\mathcal{K}_\infty)$, where $\mathcal{K}_\infty$ is a point set of class $[0, 1, \alpha + 1, q + 1]$. □

Remarks. 1. Theorem 5.1 is in fact a generalization of a similar result of De Clerck and Delanote [4] about affine line sets of antiflag type $(0, \alpha)$, $\alpha > 1$.

2. Let $\mathcal{K}_\infty$ be a point set of class $[0, 1, \alpha + 1, q + 1]$ in Π_∞. We say that $\mathcal{K}_\infty$ is *singular* if it has a *singular point*, that is, a point $p_\infty \in \mathcal{K}_\infty$ such that every line of Π_∞ through p_∞ is either completely contained in $\mathcal{K}_\infty$, or intersects $\mathcal{K}_\infty$ in the point p_∞ only. One verifies that p_∞ is a singular point of the line set of $T^*_{n-1}(\mathcal{K}_\infty)$ if and only if p_∞ is a singular point of $\mathcal{K}_\infty$.

6. Line sets of antiflag type $(0, \alpha)$

A $(0, \alpha)$-*geometry* $\mathcal{S}$ is a connected partial linear space of order (s, t) and of antiflag type $(0, \alpha)$.

If $\mathcal{S}$ is a $(0, \alpha)$-geometry fully embedded in $\mathrm{AG}(n, q)$, then clearly the line set of $\mathcal{S}$ is a connected line set of $\mathrm{AG}(n, q)$ of antiflag type $(0, \alpha)$. Conversely if $\alpha > 1$ and $\mathcal{L}$ is a connected line set of $\mathrm{AG}(n, q)$ of antiflag type $(0, \alpha)$ then one easily proves that there exists a positive integer t such that $\mathcal{S}(\mathcal{L})$ is of order $(q - 1, t)$.

So line sets of $\mathrm{AG}(n, q)$ of antiflag type $(0, \alpha)$, with $\alpha > 1$, are equivalent to $(0, \alpha)$-geometries fully embedded in $\mathrm{AG}(n, q)$. The $(0, \alpha)$-geometries with $\alpha > 1$, fully embedded in $\mathrm{AG}(n, q)$ are however classified. The most difficult part of the classification is the case $\alpha = 2$, which was completely solved by De Feyter in [7]. For an overview of the results of affine $(0, \alpha)$-geometries we refer to [2]. Only the following cases can occur.

1. Let $\mathcal{K}_\infty$ be a set of type $(1, \alpha+1)$ or of type $(0, 1, \alpha+1)$ in Π_∞. Then the linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ is a $(0, \alpha)$-geometry fully embedded in $\mathrm{AG}(n, q)$.
2. Let q be even, then the geometries HT_3 and HT_4^- are affine $(0, 2)$-geometries. The geometry HT_3 is fully embedded in $\mathrm{AG}(3, q)$, q even. The geometry HT_4^-, which is also denoted by $\mathrm{TQ}(4, q)$, is fully embedded in $\mathrm{AG}(4, q)$, q even. For a characterization theorem of the geometry HT_4^-, we refer to [1].

   In both cases, every affine plane is of type I, II or IV. So there are no planes of type III.
3. In [5], a $(0, 2)$-geometry $\mathcal{A}(O_\infty)$ fully embedded in $\mathrm{AG}(3, q)$, q even, is constructed as follows. Let O_∞ be an oval of Π_∞ with nucleus n_∞. Choose a basis such that $\Pi_\infty : X_3 = 0$, $n_\infty(1, 0, 0, 0)$ and $(0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 0) \in O_\infty$. Let f be the o-polynomial such that

   $$O_\infty = \{(\rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(0, 1, 0, 0)\},$$

   and for every affine point $p(x, y, z, 1)$ let

   $$O^p_\infty = \{(y + zf(\rho) + \rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(z, 1, 0, 0)\}.$$

   Let S_p be the set of lines through p and a point of O^p_∞. Let $\mathcal{L}$ be the union of the sets S_p, for all affine points p. If O_∞ is not a conic then $\mathcal{S}(\mathcal{L})$ is connected [5] and we put $\mathcal{A}(O_\infty) = \mathcal{S}(\mathcal{L})$. If O_∞ is a conic then $\mathcal{S}(\mathcal{L})$ consists of two connected components, both of which are projectively equivalent with the geometry HT_3 [5]. Therefore we put $\mathcal{A}(O_\infty) = \mathrm{HT}_3$ if O_∞ is a conic. In either case $\mathcal{A}(O_\infty)$ is a $(0, 2)$-geometry with $s = q - 1$, $t = q$, fully embedded in $\mathrm{AG}(3, q)$. Every affine plane is a plane of type I, II or IV. So there are no planes of type III.

4. In [6], a (0, 2)-geometry $\mathcal{I}(n, q, e)$ fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, q even, is constructed as follows. Let U be a hyperplane of $\mathrm{AG}(n, q)$. Choose a basis such that $\Pi_\infty : X_n = 0$ and $U : X_{n-1} = 0$. Let $e \in \{1, 2, \ldots, h-1\}$ be such that $\gcd(e, h) = 1$, and let φ be the collineation of $\mathrm{PG}(n, q)$ such that

$$\varphi : p(x_0, x_1, \ldots, x_{n-1}, x_n) \mapsto p^\varphi(x_0^{2^e}, x_1^{2^e}, \ldots, x_n^{2^e}, x_{n-1}^{2^e}).$$

Put $U_\infty = U \cap \Pi_\infty$ and let $\mathcal{K}_\infty$ be the set of points of U_∞ fixed by φ. Then $\mathcal{K}_\infty$ is the point set of a projective geometry $\mathrm{PG}(n-2, 2) \subseteq U_\infty$. Let $\mathcal{L}$ be the set of affine lines L such that either $L \subseteq U$ and $L \cap \Pi_\infty \in \mathcal{K}_\infty$, or L intersects U in an affine point p and L intersects Π_∞ in the point p^φ. Then $\mathcal{I}(n, q, e) = \mathcal{S}(\mathcal{L})$ is a (0, 2)-geometry with $s = q - 1$, $t = 2^{n-1} - 1$, fully embedded in $\mathrm{AG}(n, q)$.

The hyperplane U has the property that, for every affine plane π containing two intersecting lines of $\mathcal{I}(n, q, e)$, π is of type III if $\pi \subseteq U$ and π is of type IV if $\pi \nsubseteq U$. In particular, if $n = 3$, then U is the only plane of type III.

5. A dual oval with nucleus the line at infinity is a trivial (0, 2)-geometry fully embedded in $\mathrm{AG}(2, q)$, q even.
6. Any point set of $\mathrm{AG}(n, 2)$ gives rise to a trivial (0, 2)-geometry fully embedded in $\mathrm{AG}(n, 2)$.

7. The classification of connected line sets of antiflag class $[0, \alpha, q]$ in AG(3, q)

Let $\mathcal{L}$ be a line set of $\mathrm{AG}(n, q)$ of antiflag class $[0, \alpha, q]$. Without loss of generality, we may assume that $\mathcal{L}$ is not contained in a hyperplane of $\mathrm{AG}(n, q)$ and that $\mathcal{L}$ is connected. By Theorems 5.1 and 4.1, we may assume that $\mathcal{L}$ is nonsingular and that there is a plane of type IV. Hence $\alpha = 2$ and $q = 2^h$. As explained in the former section, the connected affine line sets of antiflag type $(0, \alpha)$ are classified, hence we may assume that there exists an antiflag $\{p, L\}$ such that $\alpha(p, L) = q$, i.e., we may assume that there is at least one plane of type V.

The next step towards a complete classification of affine line sets of antiflag class $[0, \alpha, q]$ is to find connected line sets or geometries fully embedded in $\mathrm{AG}(3, q)$, of antiflag type $(0, 2, q)$, $q = 2^h$, $h > 1$, such that there is at least a plane of type IV and one of type V.

Theorem 7.1. *A nonsingular connected line set of* $\mathrm{AG}(3, q)$*,* $q = 2^h$*,* $h > 1$*, of antiflag type* $(0, 2, q)$*, such that there is a plane of type* IV *and a plane of type* V*, does not exist.*

Proof. Suppose that such a set $\mathcal{L}$ does exist. Let π_1 be a plane of type V. Suppose that no plane intersecting π_1 in an affine line, is of type V. Let π be a plane parallel to but distinct from π_1. We prove that π is of type V.

Let m be the number of points of $\mathcal{S}(\mathcal{L})$ on π. Then $m > 0$ since otherwise every line of $\mathcal{L}$ is parallel to π, a contradiction. Let p be a point of $\mathcal{S}(\mathcal{L})$ in π, and let S_1 be the set of affine points of π_1 that are collinear to p in $\mathcal{S}(\mathcal{L})$. Then S_1 is a point set of type $(0, 2, q)$ in π_1. If there is an affine line $L \subseteq \pi_1$ containing q points of S_1, then the plane $\langle p, L\rangle$ is of type V, a contradiction. So S_1 is a point set of type (0, 2), and so it is a hyperoval ($S_1 \neq \emptyset$ since θ_p spans Π_∞).

By Corollary 3.2, $\mathcal{S}(\mathcal{L})$ has an order $(q - 1, t)$. Every point of $\mathcal{S}(\mathcal{L})$ in π is contained in precisely $q + 2$ lines of $\mathcal{L}$ which intersect π_1 in an affine point. Counting the lines of $\mathcal{L}$ which intersect π and π_1 in an affine point yields $m(q + 2) = q^2(t - q)$. It follows that $\frac{1}{2}q + 1 \mid t - q$. As $0 < m \leq q^2$, there are only two possibilities.

1. $t - q = \frac{1}{2}q + 1$. Then every point of $\mathcal{S}(\mathcal{L})$ in π is contained in precisely $t + 1 - (q + 2) = \frac{1}{2}q$ lines of $\mathcal{L}_\pi$. Hence $q = 4$ and π is a plane of type IV. But this contradicts $m = \frac{1}{2}q^2$.

2. $t-q=q+2$. Then every point of $\mathcal{S}(\mathcal{L})$ in π is contained in precisely $t+1-(q+2)=q+1$ lines of $\mathcal{L}_\pi$, and $m=q^2$. Hence π is a plane of type V.

It follows that every plane parallel to π_1 is of type V. But this contradicts the fact that there is a plane of type IV. We conclude that there is a plane π_2 of type V which intersects π_1 in an affine line L_0.

Let $p_\infty = L_0 \cap \Pi_\infty$. Let L be a line of $\mathcal{L}$ which intersects π_1 and π_2 in distinct affine points p_1 and p_2 respectively. We prove that the plane $\pi = \langle L, p_\infty \rangle$ is of type V. Let S_1 denote the set of affine points of π_1 that are collinear to p_2 in $\mathcal{S}(\mathcal{L})$. Then S_1 is a point set of type $(0, 2, q)$ which contains p_1 and the affine points of L_0. Hence either S_1 is the set of all affine points of π_1, or S_1 consists of the affine points of the lines L_0 and $L_1 = \langle p_1, p_\infty \rangle$. In both cases, π is a plane of type V.

Let π' be a plane through L, not parallel to the line L_0. Then π' contains three nonconcurrent lines of $\mathcal{L}$, namely L, $\pi' \cap \pi_1$ and $\pi' \cap \pi_2$. So π' is of type III, IV or V. Now applying the same reasoning as above, one can prove that for every line $L' \in \mathcal{L}_{\pi'}$, the plane $\langle L', p_\infty \rangle$ is of type V, and hence that p_∞ is a singular point of $\mathcal{L}$. But this contradicts our assumption that $\mathcal{L}$ is nonsingular. □

Theorem 7.2. *Let $\mathcal{L}$ be a connected line set of* AG$(3, q)$ *of antiflag class* $[0, \alpha, q]$, $1 < \alpha < q-1$. *Then one of the following cases occurs.*

1. $q = 2^h$, $\alpha = 2$ *and $\mathcal{L}$ is the line set of the geometry $\mathcal{A}(O_\infty)$ (we recall that* $\mathrm{HT}_3 = \mathcal{A}(O_\infty)$, *$O_\infty$ a conic).*
2. $q = 2^h$, $\alpha = 2$ *and $\mathcal{L}$ is the line set of the geometry $\mathcal{I}(3, q, e)$.*
3. *$\mathcal{L}$ is the line set of a linear representation $T_2^*(\mathcal{K}_\infty)$, with $\mathcal{K}_\infty$ a nonsingular point set of class* $[0, 1, \alpha+1, q+1]$ *in Π_∞.*
4. *$\mathcal{L}$ is the singular line set with vertex a point $p_\infty \in \Pi_\infty$ and base a planar net in an affine plane skew to p_∞.*
5. $q = 2^h$, $\alpha = 2$ *and $\mathcal{L}$ is the singular line set with vertex a point $p_\infty \in \Pi_\infty$ and base a dual oval in an affine plane skew to p_∞.*
6. *$\mathcal{L}$ is the set of all lines of* AG$(3, q)$.

Proof. This follows immediately from Theorems 4.1 and 5.1, from Section 6 and from Theorem 7.1. □

Towards the complete classification

In a forthcoming paper [3] we will provide a complete classification of connected affine partial linear spaces of antiflag class $[0, 2, q]$ that are not linear representations of point sets of class $[0, 1, 3, q+1]$ in the space at infinity Π_∞.

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